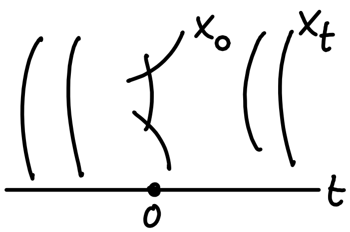
E. Ionel - 29/5/08 - Symplectic deformation and GW theory

<u>Motivation</u>: understand GW invariants and their behavior under deformations, more specifically degenerations (know: GW don't change under smooth deformations)



deformation X_t smooth symplectic for $t \neq 0$
X_0 singular

$\rightsquigarrow \lim_{t\to 0} \overline{\mathcal{M}}(X_t) = \text{“}\overline{\mathcal{M}}(X_0)\text{”}$

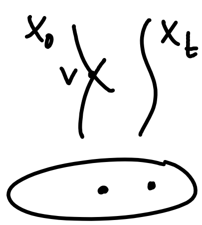
<u>problem</u>: understand which holom. curves appear as limits

<u>Q</u>: what kind of singularities should X_0 be allowed to have?

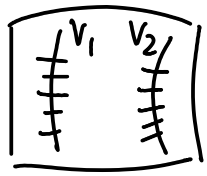
<u>Ex</u>: <u>stable degenerations</u> in alg. geom. $\rightsquigarrow$ X_0 <u>normal crossings</u>

- <u>Simplest example</u>: <u>symplectic sum</u>:

<u>Assume</u> X_0 singular, $\mathrm{sing}(X_0) = V$ smooth codim. 2 sympl. submanifold (divisor)



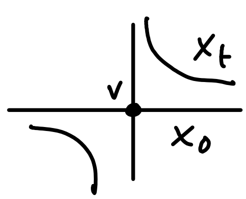
$X_0 \leftarrow \tilde{X}$ = normalization
smooth sympl. mfld containing 2 copies of V

$\tilde{X}$ 

$\longrightarrow X_0 = \tilde{X}/_{V_1 = V_2}$

with $N_{V_1} \otimes N_{V_2} \cong \mathbb{C}$.

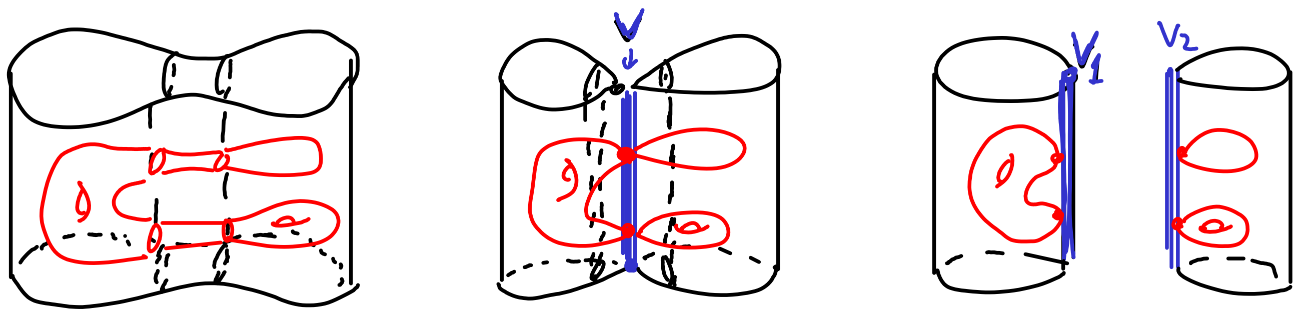
<u>Local picture</u>:



$$X_t = \{xy = t\} \subset \begin{array}{c} N_{V_1} \oplus N_{V_2} \\ \downarrow \\ V \end{array}$$

= symplectic sum of $\tilde{X}$ along $V_1 = V_2$.

- Holom. curves in $\tilde{X}$ that appear as limit of holom. curves in X_t are special: they satisfy certain matching conditions along $V_1 = V_2$



$$X_t \longrightarrow X_0 \longleftarrow \tilde{X} = X_1 \sqcup X_2$$

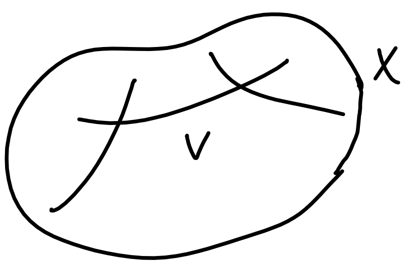
(for simplicity assume disconnected)

$\rightsquigarrow$ $\| \; GW(X_t) = GW(X_1, V_1) * GW(X_2, V_2)$ when V_i smooth

"product over $GW(V)$"

(Ionel-Parker, Jun Li)

- This is a starting pt, but need to generalize this formula to cases where V_i can be singular in order to be able to handle more interesting degenerations (e.g. iterate the above: if we break (X,V) into 2 pieces, get 2 divisors inside the pieces)

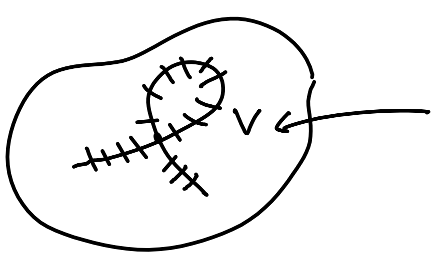
$\Rightarrow$ 

Need to define:

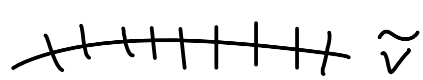
Rel. GW invariant of X_i smooth rel. to a normal crossing divisor V_i?

and study its behavior under deformation?

NB:



$\tilde{V}$ = normalization of V – smooth, symplectic

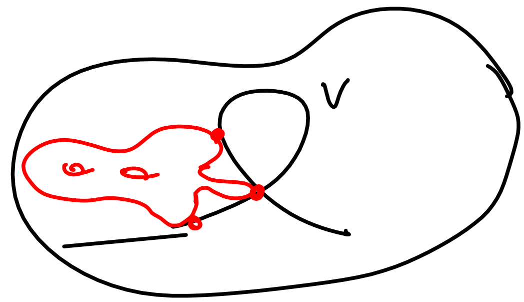


⇒ <u>Step 1</u>: define relative invts $GW(X,V)$ for V = normal crossing divisor

define $\overline{\mathcal{M}}(X,V)$ & show it carries a fundamental cycle?

<u>Step 2</u>: prove a degeneration formula ("Mayer-Vietoris formula")

Consider maps with marked points and incidence conditions on strata of V at the marked points



$\mathcal{M}(X,V)$ = holom. maps into (X,V)

$f: \Sigma \to X$

marked points $x_i \in \Sigma$ record $f^{-1}(V)$

Each marked point x has <u>multiplicity information</u> $s(x)$:

if $f(x) \in V^{(k)}$ codim. k stratum of k,

the corresponding mult. data is $s(x) = (s_1, .., s_k)$

s.t. in local coordinates $f_j(z) = a_j z^{s_j}$ + higher order terms

$$\begin{cases} a_j \neq 0 & \text{leading term coefficient} \\ s_j = & \text{multiplicity of contact to } j^{\text{th}} \text{ branch of } V \end{cases}$$

Then we have evaluation map $ev_x : \mathcal{M}(X,V) \to V^{(k)}$, $k = |s(x)|$

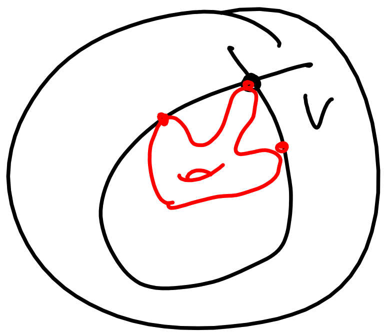
(convention $V^{(0)} = X$)

& stabilization map $st: \mathcal{M}(X,V) \to \overline{\mathcal{M}}_{g,\ell(s)}$

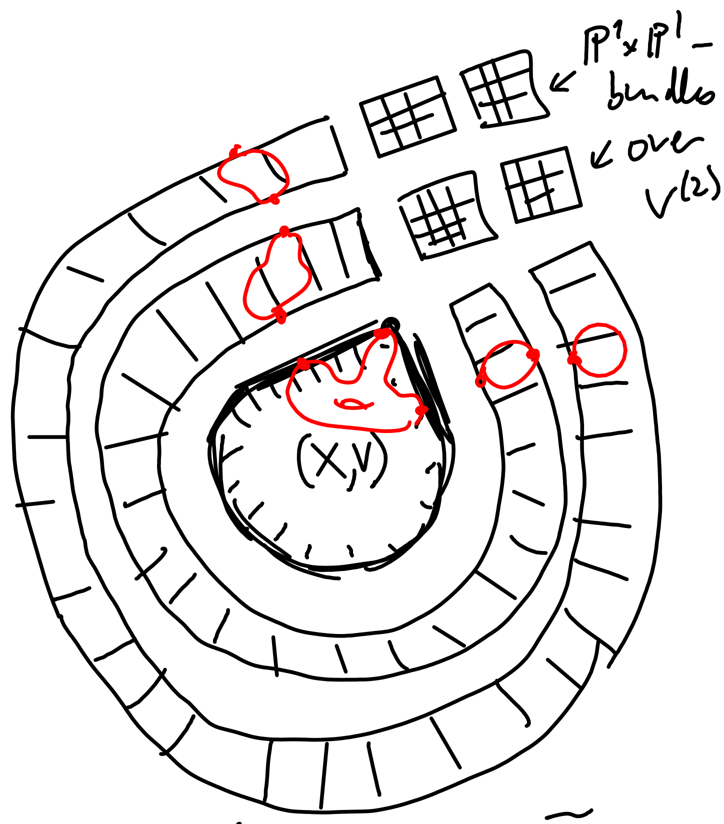
Need to <u>compactify $\overline{\mathcal{M}}(X,V)$</u> & prove it carries a fund. cycle

<u>Point</u>: what if, in a sequence of curves, some limit component collapses onto $V^{(k)}$? → need to rescale appropriately the normal directions, otherwise dimensions of strata will be wrong

Ie: rescale normal to V:



$\longrightarrow$ rescaling limit

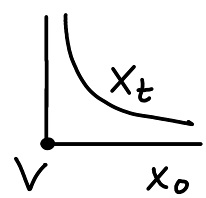


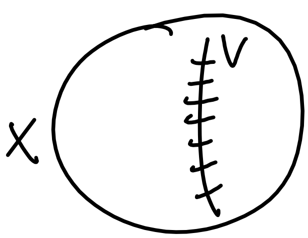
$\mathbb{P}^1$-bundles over $\tilde{V}$

with matching conditions
- same intersection # with V, same multiplicities

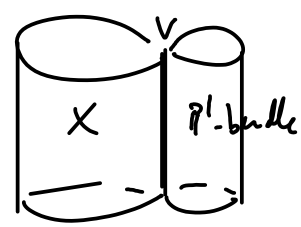
Not all curves that satisfy these matching conditions appear as limits of curves in X_t $\Rightarrow$ need enhanced matching conditions....

Ex: target := $X_t = \{xy = t\}$

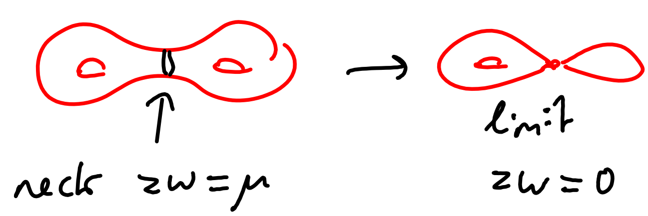




$\rightsquigarrow$ rescaling limit: $X \# \mathbb{P}^1$-bundle over V



Now, domain of curves in X_t:



Then: in limit $x(z) = a z^s + \dots$
$y(w) = b w^s + \dots$ same contact orders s

and need: $xy = ab(zw)^s + \dots = ab\mu^s + \dots = t$

$\rightsquigarrow$ condition on leading coeffts a & b !! namely $\forall t\ \exists \mu / ab\mu^s = t$

When there are several breaking points, need t = same at all pts!

- Similarly in higher codim: when domain breaks,

$$\left.\begin{aligned} x_j(z) &= a_j z^{s_j} + \dots \\ y_j(w) &= b_j w^{s_j} + \dots \end{aligned}\right\}$$ need matching condition on coefficients so $a_j b_j \mu^{s_j}$ are compatible !!

cotangent line $\mathcal{L}_x = T^*_x\Sigma$

$\Rightarrow$ extra information: lead coefficients $a_i \in ev^* N_{V_j} \otimes \mathcal{L}_x$ must be kept track of in "enhanced evaluation map"

$\rightsquigarrow$ get: $\widetilde{ev}_x : \overline{\mathcal{M}(X, V)} \longrightarrow \mathbb{P}_{s(x)}(N_{V^{(k)}})$

projectivization of the leading coeffts lives in fiber of this bundle over $f(x) \in V^{(k)}$. ✓

$\rightsquigarrow$ relative GW invts by pulling back classes from $\begin{cases} \widetilde{ev}_x & \text{enhanced eval.} \\ st & \text{stabilization} \end{cases}$

$\rightsquigarrow$ generalized symplectic sum formula for these invariants

(involves fiber product of relative moduli spaces for the various pieces over enhanced compatibility condition at marked points from enhanced evaluation maps)

⚠ $\widetilde{ev}_x$ only sees $\mathbb{P}$(leading coeffts), need a bit more to formulate the compatibility condition ...